

## A UNIQUE COMMON FIXED POINT RESULT IN CONE METRIC SPACE UNDER GENERALIZED ALTERING DISTANCE FUNCTIONS

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**ABSTRACT.** In the present paper we establish a unique common fixed point result for two-self mappings of a cone metric space involving two generalised altering distance functions. The result obtained generalizes the fixed point result of Choudhury (2005) from metric space to cone metric space.

### 1. INTRODUCTION

On the theory of fixed points for mappings on cone valued metric space, we have some literature since last four decades. Recently the concept of altering distance function is used by Khan et al (1984), Sastry et al. (1998, 1999), Choudhury (2005) etc. as control function, which has given a new direction to the fixed point theory on metric spaces. In the present work we use this control function on cone metric spaces and obtain a common fixed point result. Before going to our main result, we give here some related preliminary definitions and concepts.

Let  $E$  be a real Banach space and  $\theta$  is the zero of the Banach space  $E$ . Let  $P$  be a subset of  $E$ .  $P$  is called a cone if

- (1)  $P$  is closed, non-empty and  $P \neq \{\theta\}$
- (2)  $ax + by \in P$  for all  $x, y \in P$  and non-negative real numbers  $a, b$
- (3)  $P \cap (-P) = \{\theta\}$

For a given cone  $P$  we can define a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ .  $x < y$  will stand for  $x \leq y$  and  $x \neq y$ , while  $x << y$  will stand for  $y - x \in \text{int}P$ ; where  $\text{int}P$  denotes the interior of  $P$ .  $x \leq y$  is same as  $y \geq x$  and  $x << y$  is same as  $y >> x$ .

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A cone  $P$  is called normal if there is a number  $K > 0$  such that for all  $x, y \in E$ ,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of  $P$ . The cone is called regular if every increasing and bounded above sequence  $x_n$  in  $E$  is convergent. Equivalently the cone  $P$  is regular if and only if every decreasing and bounded below sequence is convergent.

**Definition 1.1:** Let  $X$  be a non-empty set. Suppose the mapping

$$d : X \times X \rightarrow E \text{ satisfies :}$$

- (1)  $\theta \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = \theta$  if and only if  $x = y$
- (2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .
- (3)  $d(x, y) \leq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is called a cone metric on  $X$  and  $(X, d)$  is called a cone metric space.

**Definition 1.2:** Let  $(X, d)$  be a cone metric space,  $x_n$  a sequence in  $X$  and  $x \in X$ . For every  $c \in E$  with  $\theta << c$ ; we say that  $x_n$  is:

- (1) a Cauchy sequence if there is a natural number  $N$  such that for all  $n, m > N$ ;  $d(x_n, x_m) << c$
- (2) convergent to  $x$  if there is a natural number  $N$  such that for all  $n > N$ ;  $d(x_n, x) << c$  for some  $x \in X$ .

**Definition 1.3**  $(X, d)$  is called a complete cone metric space if every Cauchy sequence in  $X$  is convergent.

## 2. SOME BASIC RESULTS

**Lemma 2.1** ([4]). Let  $(X, d)$  be a cone metric space,  $P$  be a normal cone with normal constant  $K$ . Let  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  converges to  $x$  if and only if  $d(x_n, x) \rightarrow \theta$  as  $n \rightarrow \infty$ .

**Lemma 2.2** ([4]). Let  $(X, d)$  be a cone metric space.  $P$  be a normal cone with normal constant  $K$ . Let  $\{x_n\}$  be a sequence in  $X$ . If  $x_n \rightarrow x$  as well as  $x_n \rightarrow y$  then  $x = y$ . That is the limit of  $\{x_n\}$  is unique.

**Lemma 2.3** ([4]). Let  $(X, d)$  be a cone metric space and  $\{x_n\}$  be a sequence in  $X$ . If  $\{x_n\}$  converges to  $x$ , then  $\{x_n\}$  is a Cauchy sequence.

**Lemma 2.4([4]).** Let  $(X, d)$  be a cone metric space.  $P$  be a normal cone with normal constant  $K$ . Let  $\{x_n\}$  be a sequence in  $X$ . Then  $\{x_n\}$  is a Cauchy sequence if and only if  $d(x_n, x_m) \rightarrow \theta$  as  $(n, m \rightarrow \infty)$ .

**Lemma 2.5:** Let  $E$  be a real Banach Space and  $P$  be a cone. Then  $\theta \notin \text{int}P$ .

**Proof :** Let  $0 \in \text{int}P$ . Then there exist some  $c(\neq \theta) \in P$  such that  $c << \theta$ .

$$\text{i.e., } \theta - c \in P$$

$$\text{i.e., } c \in -P, \text{ So } c \in P \cap (-P)$$

$$\text{i.e., } c = \theta, \text{ a contradiction.}$$

**Lemma 2.6:** Let  $E$  be a real Banach space and  $P$  be a cone then  $P$  must be equal to the set  $\{x \in E : \theta \leq x\}$ , where the partial order  $\leq$  is with respect to the cone  $P$ .

**Proof :** Let  $x \in P$  then  $x - \theta \in P$ . This implies that  $\theta \leq x$ . If  $\theta \leq x$ , then  $x - \theta \in P$ , i.e.,  $x \in P$ . So  $x \in P$  if and only if  $\theta \leq x$ .

Case 1: If  $P = E$  then  $(-P) = -E = E = P$ . So  $P \cap (-P) = P \neq \{\theta\}$ , a contradiction. So  $P$  must be a proper subset of  $E$ .

Case 2: If  $P = \{x \in E : \theta \leq x \leq t\}$  for some  $t \geq \theta$  belonging to  $E$ . i.e.,  $P$  has a upper bound. Then for some real number  $y$  such that  $y > 1$ , We have  $t(y - 1) \in P$  and  $ty \in P$

$$\text{i.e., } ty - t \in P, \text{ because } t \in P \text{ and } 0 < y - 1.$$

$$\text{i.e., } t \leq ty$$

$$\text{Now } (ty - t) = t(y - 1) \neq \theta, \text{ because } t \neq \theta \text{ and } y - 1 \neq 0.$$

$$\text{i.e., } t \neq ty$$

$$\text{i.e., } t \neq ty \text{ and } t \leq ty.$$

$$\text{i.e., } t << ty \text{ and } ty \in P. \text{ This is again a contradiction.}$$

So  $P$  must be a set of type  $\{x \in E : \theta \leq x\}$ , where the partial order  $\leq$  is with respect to the cone  $P$ .

### 3. MAIN RESULTS

**Definition 3.1:** A continuous function  $\psi : P \rightarrow P$  is said to be an altering distance function if it satisfies :

- (1)  $\psi$  is monotone increasing in the sence of norm, i.e.  $\|x\| < \|y\|$  implies  $\|\psi(x)\| < \|\psi(y)\|$  and
- (2)  $\psi(x) = \theta$  if and only if  $x = \theta$ .

We can motivate to define generalized altering distance function like:

**Definition 3.2:** A continuous function  $\psi : P^3 \rightarrow P$  is said to be a generalized altering distance function if

- (1)  $\psi$  is monotone increasing in all the three variables in the sense of norm, and
- (2)  $\psi(x, y, z) = \theta$  if and only if  $x = y = z = \theta$ .

**Theorem 3.1:** Let  $P$  be the normal cone. Let  $(X, d)$  be a complete cone metric space and  $S$  and  $T$  be two self- mappings such that the following inequality is satisfied:

$$(1) \quad \begin{aligned} \|\varphi_1(d(Sx, Ty))\| &\leq \|\psi_1(d(x, y), d(x, Sx), d(y, Ty))\| \\ &\quad - \|\psi_2(d(x, y), d(x, Sx), d(y, Ty))\| \end{aligned}$$

where  $\psi_1$  and  $\psi_2$  are generalized altering distance functions and

$$\varphi_i(x) = \psi_i(x, x, x), \quad \text{where } i = 1, 2.$$

Then  $S$  and  $T$  have a unique common fixed point.

**Proof:** Let  $x_0 \in X$  be an arbitrary point. Define

$$(2) \quad x_{2n+1} = Sx_{2n} \quad \text{for } n = 0, 1, 2, 3 \dots$$

and

$$(3) \quad x_{2n+2} = Tx_{2n+1}, \quad \text{for } n = 0, 1, 2, 3 \dots$$

Let  $a_n = d(x_n, x_{n+1})$ . Putting  $x = x_{2n}$  and  $y = x_{2n+1}$  in (1), we get for all  $n = 0, 1, 2 \dots$

$$\begin{aligned} &\|\varphi_1(d(Sx_{2n}, Tx_{2n+1}))\| \\ &\leq \|\psi_1(d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}))\| \\ &\quad - \|\psi_2(d(x_{2n}, x_{2n+1}), d(x_{2n}, Sx_{2n}), d(x_{2n+1}, Tx_{2n+1}))\| \end{aligned}$$

i.e.,

$$\begin{aligned} &\|\varphi_1(d(x_{2n+1}, x_{2n+2}))\| \\ &\leq \|\psi_1(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}))\| \\ &\quad - \|\psi_2(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}))\| \end{aligned}$$

i.e.,

$$(4) \quad \begin{aligned} & \|\varphi_1(a_{2n+1})\| \\ & \leq \|\psi_1(a_{2n}, a_{2n}, a_{2n+1})\| - \|\psi_2(a_{2n}, a_{2n}, a_{2n+1})\| \end{aligned}$$

If  $\|a_{2n}\| < \|a_{2n+1}\|$ , then  $0 < \|\psi_2(a_{2n}, a_{2n}, a_{2n+1})\|$ . This implies

$$\begin{aligned} & \|\varphi_1(a_{2n+1})\| < \|\psi_1(a_{2n}, a_{2n}, a_{2n+1})\| \\ & < \|\psi_1(a_{2n+1}, a_{2n+1}, a_{2n+1})\| \end{aligned}$$

i.e.,

$$\|\varphi_1(a_{2n+1})\| < \|\varphi_1(a_{2n+1})\|$$

This is a contradiction. So for  $n = 0, 1, 2 \dots$

$$(5) \quad \|a_{2n+1}\| \leq \|a_{2n}\|$$

Putting  $x = x_{2n}$  and  $y = x_{2n-1}$  in (1) we obtain

$$(6) \quad \begin{aligned} & \|\varphi_1(a_{2n})\| \\ & \leq \|\psi_1(a_{2n-1}, a_{2n-1}, a_{2n})\| - \|\psi_2(a_{2n-1}, a_{2n-1}, a_{2n})\| \end{aligned}$$

Similarly we can show that,

$$(7) \quad \|a_{2n+2}\| \leq \|a_{2n+1}\| \quad \text{for all } n = 0, 1, 2, 3 \dots$$

From (5) and (7), we obtain,

$$(8) \quad \|a_{n+1}\| \leq \|a_n\| \quad \text{for all } n = 0, 1, 2, 3 \dots$$

Then from (4) and (6) we obtain, for all  $n = 0, 1, 2 \dots$

$$\|\varphi_1(a_{n+1})\| \leq \|\varphi_1(a_n)\| - \|\varphi_2(a_{n+1})\|$$

i.e.,

$$\|\varphi_2(a_{n+1})\| \leq \|\varphi_1(a_n)\| - \|\varphi_1(a_{n+1})\|$$

Summing up from 0 to  $\infty$  we obtain

$\sum \|\varphi_2(a_{n+1})\| \leq \|\varphi_1(a_0)\| < \infty$ , i.e.,  $\sum \|\varphi_2(a_{n+1})\|$  is a real convergent series. So it's  $n^{th}$  term must goes to zero as  $n$  tends to infinity. i.e.,  $\|\varphi_2(a_n)\| \rightarrow 0$  as  $n \rightarrow \infty$

Now from (8), we can say that  $\{\|a_n\|\}$  is bounded monotone decreasing sequence of positive real numbers, so by Bolzano-Weierstrass Theorem,

$\|a_n\| \rightarrow a$  as  $n \rightarrow \infty$ , for some real number  $a$ .

Now we will prove that  $a_n \rightarrow \theta$ . We have

$$(9) \quad \|\varphi_2(a_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

i.e.,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\varphi_2(a_n)\| &= 0 \\ \Rightarrow \|\lim_{n \rightarrow \infty} \varphi_2(a_n)\| &= 0 \\ \Rightarrow \|\varphi_2(\lim_{n \rightarrow \infty} a_n)\| &= 0 \\ \Rightarrow \varphi_2(\lim_{n \rightarrow \infty} a_n) &= \theta \\ \Rightarrow \lim_{n \rightarrow \infty} a_n &= \theta \end{aligned}$$

i.e.,

$$(10) \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \theta$$

We now prove that  $\{x_n\}$  is a Cauchy sequence. In the view of (10) it is sufficient to prove that  $\{x_{2r}\} \subset \{x_n\}$  is Cauchy sequence. If  $\{x_{2r}\}$  is not a Cauchy sequence, then given

$\epsilon \in \text{int}P$  such that  $\|\epsilon\| > 0$  we can find monotone increasing sequences of natural numbers  $\{2m(k)\}$  and  $\{2n(k)\}$  such that, for  $n(k) > m(k)$ ,

$$(11) \quad \begin{aligned} \|d(x_{2m(k)}, x_{2n(k)})\| &> \|\epsilon\| \quad \text{and} \\ \|d(x_{2m(k)}, x_{2n(k)-1})\| &< \|\epsilon\| \end{aligned}$$

Then

$$\begin{aligned} \|\epsilon\| &< \|d(x_{2m(k)}, x_{2n(k)})\| \\ &\leq \|d(x_{2m(k)}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)})\| \\ &\leq \|d(x_{2m(k)}, x_{2n(k)-1})\| + \|d(x_{2n(k)-1}, x_{2n(k)})\| \\ &< \|\epsilon\| + \|d(x_{2n(k)-1}, x_{2n(k)})\| \end{aligned}$$

Taking  $k \rightarrow \infty$  in the above inequality we obtain

$$(12) \quad \lim_{k \rightarrow \infty} \|d(x_{2m(k)}, x_{2n(k)})\| = \|\epsilon\|$$

Now, for all  $k = 1, 2, \dots$

$$\begin{aligned} &\|d(x_{2n(k)+1}, x_{2m(k)})\| \\ &\leq \|d(x_{2n(k)+1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2m(k)})\| \end{aligned}$$

i.e.,

$$(13) \quad \begin{aligned} &\|d(x_{2n(k)+1}, x_{2m(k)})\| \\ &\leq \|d(x_{2n(k)+1}, x_{2n(k)})\| + \|d(x_{2n(k)}, x_{2m(k)})\| \end{aligned}$$

and

$$\begin{aligned} & \|d(x_{2n(k)}, x_{2m(k)})\| \\ & \leq \|d(x_{2n(k)}, x_{2n(k)+1}) + d(x_{2n(k)+1}, x_{2m(k)})\| \end{aligned}$$

i.e.,

$$\begin{aligned} (14) \quad & \|d(x_{2n(k)}, x_{2m(k)})\| \\ & \leq \|d(x_{2n(k)}, x_{2n(k)+1})\| + \|d(x_{2n(k)+1}, x_{2m(k)})\| \end{aligned}$$

Taking  $k \rightarrow \infty$  in the above inequalities and using (10) and (12) we obtain

$$\lim_{k \rightarrow \infty} \|d(x_{2n(k)+1}, x_{2m(k)})\| \leq \|\epsilon\|$$

and

$$\|\epsilon\| \leq \lim_{k \rightarrow \infty} \|d(x_{2n(k)+1}, x_{2m(k)})\|$$

i.e.,

$$(15) \quad \lim_{k \rightarrow \infty} \|d(x_{2n(k)+1}, x_{2m(k)})\| = \|\epsilon\|$$

Now for all  $k = 1, 2, \dots$

$$\begin{aligned} & \|d(x_{2n(k)}, x_{2m(k)-1})\| \\ & \leq \|d(x_{2n(k)}, x_{2m(k)}) + d(x_{2m(k)}, x_{2m(k)-1})\| \\ & \leq \|d(x_{2n(k)}, x_{2m(k)})\| + \|d(x_{2m(k)}, x_{2m(k)-1})\| \end{aligned}$$

and

$$\begin{aligned} & \|d(x_{2n(k)}, x_{2m(k)})\| \\ & \leq \|d(x_{2n(k)}, x_{2m(k)-1}) + d(x_{2m(k)-1}, x_{2m(k)})\| \\ & \leq \|d(x_{2n(k)}, x_{2m(k)-1})\| + \|d(x_{2m(k)-1}, x_{2m(k)})\| \end{aligned}$$

Taking  $k \rightarrow \infty$  in the above inequalities and using (10) and (15) we obtain

$$(16) \quad \lim_{k \rightarrow \infty} \|d(x_{2n(k)}, x_{2m(k)-1})\| = \|\epsilon\|$$

Now putting  $x = x_{2n(k)}$  and  $y = x_{2m(k)-1}$  in (1) we obtain for all  $k \in N$

$$\begin{aligned} & \|\varphi_1(d(x_{2n(k)+1}, x_{2m(k)}))\| \\ & \leq \|\psi_1(d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2n(k)+1}), \\ & \quad (x_{2m(k)-1}, x_{2m(k)}))\| \\ & \quad - \|\psi_2(d(x_{2n(k)}, x_{2m(k)-1}), d(x_{2n(k)}, x_{2n(k)+1}), \\ & \quad d(x_{2m(k)+1}, x_{2m(k)}))\| \end{aligned}$$

Taking  $k \rightarrow \infty$  in the above inequality and using (10), (15) and (16) with the continuity and monotonicity of  $\psi_1$  and  $\psi_2$  we have,

$$(17) \quad \|\varphi_1(\epsilon)\| \leq \|\psi_1(\epsilon, 0, 0)\| - \|\psi_2(\epsilon, 0, 0)\| < \|\varphi_1(\epsilon)\|$$

This is due to the fact that  $\psi_1$  and  $\psi_2$  are monotone increasing in all the three variables in the sense of norm and  $\psi(x, y, z) = \theta$  if and only if  $x = y = z = \theta$ . So if we assume that  $\{x_{2r}\}$  is not Cauchy sequence then we finally achieve a contradiction. That means  $\{x_{2r}\}$  is a Cauchy sequence hence by (10),  $\{x_n\}$  is also a Cauchy sequence.

As  $(X, d)$  is a complete cone metric space  $\{x_n\}$  will converge to some point  $z$  (say) belonging to  $X$ .

Putting  $x = x_{2n}$  and  $y = z$  in (1) we obtain for all  $n \in N$

$$\begin{aligned} & \|\varphi_1(d(x_{2n+1}, Tz))\| \\ & \leq \|\psi_1(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, Tz))\| \\ & \quad - \|\psi_2(d(x_{2n}, z), d(x_{2n}, x_{2n+1}), d(z, Tz))\| \end{aligned}$$

Taking  $n \rightarrow \infty$  in the above inequality, we obtain by using (9) and  $x_n \rightarrow z$  as  $n \rightarrow \infty$  with continuity of  $\psi_1$  and  $\psi_2$ , we get,

$$\|\varphi_1(d(z, Tz))\| \leq \|\psi_1(\theta, \theta, d(z, Tz))\| - \|\psi_2(\theta, \theta, d(z, Tz))\|$$

If  $d(z, Tz) \neq \theta$  then  $0 < \|\psi_2(d(\theta, \theta, d(z, Tz))\|$

So

$$\begin{aligned} \|\varphi_1(d(z, Tz))\| & < \|\psi_1(\theta, \theta, d(z, Tz))\| \\ & < \|\psi_1(d(z, Tz), d(z, Tz), d(z, Tz))\| \\ & = \|\varphi_1(d(z, Tz))\| \end{aligned}$$

i.e.,  $\|\varphi_1(d(z, Tz))\| < \|\varphi_1(d(z, Tz))\|$ .

This is a contradiction.

i.e.,  $(d(z, Tz))$  must be equal to  $\theta$ .

i.e.,  $Tz = z$ .

Similarly we can prove that  $Sz = z$ .

i.e.,  $z$  is a common fixed point of  $S$  and  $T$ .

Uniqueness : Let  $w(\neq z)$  be another common fixed point of  $S$  and  $T$ . Then  $d(z, w) \neq \theta$  and

$$\begin{aligned} \|\varphi_1(d(Sz, Tw))\| & = \|\varphi_1(d(z, w))\| \\ & \leq \|\psi_1(d(z, w), \theta, \theta)\| - \|\psi_2(d(z, w), \theta, \theta)\| \\ & < \|\varphi_1(d(z, w))\| \end{aligned}$$

This is a contradiction. So  $d(z, w) = \theta$ . i.e.,  $z = w$ .

**Corollary 3.1:** Let  $(X, d)$  is a complete cone metric space and  $S, T : X \rightarrow X$  which satisfy :

$$(18) \quad \begin{aligned} 3\|d(Sx, Ty)\| &\leq \|d(x, y) + d(x, Sx) + d(y, Ty)\| \\ &\quad - k \|\max\{d(x, y), d(x, Sx), d(y, Ty)\}\| \end{aligned}$$

where  $k$  is a positive real number, then  $S$  and  $T$  have a unique common fixed point.

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